A short introduction to the theory of inverted dice sums

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Published at https://www.simonjensen.com/texts/#mathematics, July 2023.

1. Background

This paper discusses the concept of *inverted dice sums*.[1] Even though the term *inverted dice roll* is sometimes used synonymously, it is a *theoretical* construction. No special dice or acrobatic ways of throwing dice are required. An inverted dice sum is a way of *interpreting* a roll of dice, and most often such roll is performed with normal six-faced dice, i.e., *standard dice*. A standard die is a *cube* with each of its sides marked with a unique number of dots (pips, points, spots) from ⚀ to ⚅, such that the sum of the dots on two opposite sides is seven.[2] The *value* of a die is the number of dots facing *upwards* after a roll.

A regular *dice sum* is the sum of the values of all dice included in a roll, i.e., the sum of all values that *are present* in a roll. A roll with *one* single standard die produces a sum with only one addend, so the sum is either 1, 2, 3, 4, 5, or 6. The dice sum in a roll with *two* standard dice (in short, the *sum of two dice*) ranges from 2 to 12, where 7 is most likely to occur (with a probability of $1/6$).

In general, the sum of n standard dice ranges from n to $6n$, where $\lfloor \frac{7}{2} n \rfloor$ is most likely to occur.[3]

An *inverted dice sum* is the sum of all the values that *are not present* in a roll. This is not to be confused with the reversed value you get, when counting the dots on the sides *facing downwards* for one or more dice.

Let $D = \{1, 2, 3, 4, 5, 6\}$ be the set of all possible dice values. For each roll, let S be the set of unique *shown* values (i.e., distinct values that *are present* in the roll).[4] The inverted sum is the sum of all elements in $D \setminus S$.

Since $1 + 2 + 3 + 4 + 5 + 6 = 21$, an inverted dice sum can be calculated by subtracting all values in S from 21. A set does not have duplicate elements, so we only subtract each unique value once. Let's look at a few examples:

Roll	Sum	Inverted sum	Explanation of the inverted sum
⚁	2	**19**	⚀ + ⚂ + ⚃ + ⚄ + ⚅ is **19**. Also, $21 -$ ⚁ $=$ **19**.[5]
⚁ ⚁	4	**19**	As above, ⚁ is the only value present, and $21 - 2 =$ **19**.
⚁ ⚄	7	**14**	⚀ + ⚂ + ⚃ + ⚅ is **14**. Also, $21 -$ ⚁ $-$ ⚄ $=$ **14**.
⚁ ⚄ ⚄	12	**14**	As above, 2 and 5 are the only values in S, and $21 - 7 =$ **14**.
⚁ ⚄ ⚅	13	**8**	⚀ + ⚂ + ⚃ is **8**. Also, $21 - 2 - 5 - 6 =$ **8**.
⚀ ⚀ ⚀	3	**20**	1 is the only value shown, and $21 - 1 =$ **20**.
⚅ ⚅ ⚅	18	**15**	6 is the only value shown, and $21 - 6 =$ **15**.
⚀ ⚁ ⚁ ⚂	8	**15**	⚃ + ⚄ + ⚅ is **15**. Also, $21 - (1 + 2 + 3) =$ **15**.
⚀ ⚁ ⚂ ⚃	10	**11**	⚄ + ⚅ is **11**. Also, $21 - (1 + 2 + 3 + 4) =$ **11**.
⚁ ⚂ ⚃ ⚄ ⚅	20	**1**	⚀ is the only value *not* shown, so $D \setminus S = \{\mathbf{1}\}$.
⚀ ⚁ ⚂ ⚃ ⚄	15	**6**	**6** is the only value *not* shown.
⚀ ⚀ ⚀ ⚀ ⚀	5	**20**	⚀ is the only value shown, and $21 - 1 =$ **20**.
⚀ ⚀ ⚀ ⚀ ⚀ ⚀	6	**20**	As above, 1 is the only value shown, and $21 - 1 =$ **20**.
⚀ ⚁ ⚂ ⚃ ⚄ ⚅	21	**0**	$S = \{1, 2, 3, 4, 5, 6\}$, so $D \setminus S = \emptyset$, and $21 - 21 =$ **0**.
⚁ ⚂ ⚃ ⚄ ⚅ ⚅	26	**1**	$S = \{2, 3, 4, 5, 6\}$, so $D \setminus S = \{\mathbf{1}\}$, and $21 - 20 =$ **1**.

Table 1. Examples of dice rolls with their regular sums and inverted sums.

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[1] For information about the game *Inverted Dice*™, see https://www.simonjensen.com/InvertedDice. It might be a good starting point.

[2] The concept of inverted dice sums works for all types of sequentially numbered f-faced dice, where $f > 1$, regardless of their shape. Various extensions of the concept can be made to deal with other types of dice.

[3] $\lfloor\ \rfloor$ denote the *floor function*, so for even n, we get $\frac{7n}{2}$. For odd n, we get $\frac{7n-1}{2}$. See https://mathworld.wolfram.com/Dice.html for more information about the probabilities involved here.

[4] A more formal definition of S is given in section 2.5 of this paper.

[5] Strictly speaking, the dots themselves are not the *value* of a die. But it is pedagogical to associate ⚀ with 1, ⚁ with 2, and so on.

As indicated in Table 1, an inverted dice sum does not depend on *the number of dice* the same way a regular sum does. For instance, both ⚄⚄ and ⚄⚄⚄⚄⚄⚄ has the inverted sum 16, since for both these rolls, S equals $\{5\}$, so $D \setminus S = \{1, 2, 3, 4, 6\}$, which gives us the inverted sum

$$\sum_{k \in D \setminus S} k = 1 + 2 + 3 + 4 + 6 = 16.$$

However, the *possible* inverted sums depend on the number of dice, see Table 2 below.

2. The six-faced case

The discussion in this section concerns 6-faced (6-sided) dice. As before, let $n > 0$ be the number of dice in a roll. Let $\sigma(n)$ be the number of possible distinct *regular* sums and $\dot{\sigma}(n)$ be the number of possible distinct *inverted* sums for each roll.[6]

n	Possible regular sums	$\sigma(n)$	Possible inverted sums	$\dot{\sigma}(n)$
1	1 to 6	6	15 to 20	6
2	2 to 12	11	10 to 20	11
3	3 to 18	16	6 to 20	15
4	4 to 24	21	3 to 20	18
5	5 to 30	26	1 to 20	20
$\boldsymbol{n > 5}$	$\boldsymbol{n}$ **to** $\boldsymbol{6n}$	$\boldsymbol{5n + 1}$	**0 to 20**	**21**

Table 2. Possible distinct *regular sums* and distinct *inverted sums* of n six-faced dice.

For rolls with n dice, the number of possible *inverted* sums is given by

$$\dot{\sigma}(n) = \sum_{k=1}^{6} k \cdot [n + k > 6]$$

which is 21 for all $n \geq 6$, while the number of possible *regular* sums is given by $\sigma(n) = 5n + 1$ which has no upper bound as n increases.[7] Inverted sums of n dice are ranging from $21 - \dot{\sigma}(n)$ to 20 (regular sums are ranging from n to $6n$, as mentioned earlier). If (and only if) $n = 1$ or $n = 2$, we have $\dot{\sigma}(n) = \sigma(n)$.

Again, let $D = \{1, 2, 3, 4, 5, 6\}$ be the set of possible dice values. Let R be a *multiset* representing each *distinct* roll and let $S = R \cap D$ be the *set* of distinct *shown* values in each roll (as before). In other words, S is the *support* of R in the *universe* D and the *cardinality* of R is n.[8] Neither R nor S can be empty. Elements of R are listed in non-decreasing order, occasionally with the dice notation introduced above. As an example, the roll ⚅⚁⚄⚁ is expressed as $R = \{2, 2, 5, 6\} =$ ⚁⚁⚄⚅. Brackets are omitted when using dice symbols.

Let $\Sigma(R)$ be the regular sum and let $\dot{\Sigma}(R)$ be the inverted sum for each roll R. Then, we have the trivial identities $\dot{\Sigma}(R) = \dot{\Sigma}(S) = \Sigma(D \setminus S) = \Sigma(D) - \Sigma(S) = 21 - \Sigma(S)$.

The intersection of the *images* of the functions $\dot{\Sigma}$ and Σ is empty when $n = 1$ or $n > 20$, since $\dot{\Sigma}(R) \leq 20$ and $\Sigma(R) \geq n$ for all R, and $n = 1$ is a special case. In other words, for $n = 2, 3, \ldots, 20$, there exist rolls R_1 and R_2 such that $\dot{\Sigma}(R_1) = \Sigma(R_2) = x$, where $n \leq x \leq 20$. In fact, for $n \in \{2, 3\}$, we have $21 - \dot{\sigma}(n) \leq x \leq 20$. For instance, with the rolls $R_1 =$ ⚃⚄⚅ and $R_2 =$ ⚁⚁⚁ we have $\dot{\Sigma}(R_1) = \Sigma(R_2) = 6 = 21 - \dot{\sigma}(3)$.

The probability of a roll of n dice resulting in a particular regular sum x is notated as $\Pr(n, \Sigma(R) = x)$, and the probability of an inverted sum x is notated as $\Pr(n, \dot{\Sigma}(R) = x)$. As an example, $\Pr\left(5, \dot{\Sigma}(R) = 18\right) = \frac{31}{7776}$. In other words, out of the 7776 possible outcomes of a roll with 5 dice, there are 31 with the inverted sum 18.[9]

For the regular sum probabilities, and with the notation introduced above, we have the trivial identities $\Pr(n, \Sigma(R) < n) = 0$, $\Pr(n, \Sigma(R) > 6n) = 0$, and $\Pr(n, \Sigma(R) = n) = \Pr(n, \Sigma(R) = 6n) = \frac{1}{6^n}$.

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[6] The overdot used in $\dot{\sigma}$ should not be confused with the usage of that diacritic in other texts. It has *nothing* to do with derivatives.

[7] [ ] is the Iverson bracket. Mixing algebraic and logical expressions like this might seem unnecessarily unorthodox at first sight, but in combination with the sigma notation, this approach has strong benefits.

[8] See https://en.wikipedia.org/wiki/Multiset for an explanation of the terminology used here.

[9] Those 31 possible outcomes correspond to the situations where the multiset R equals $\{1, 1, 1, 1, 2\}$ or $\{1, 1, 1, 2, 2\}$ or $\{1, 1, 2, 2, 2\}$ or $\{1, 2, 2, 2, 2\}$ or $\{3, 3, 3, 3, 3\}$. This means that S is either $\{1, 2\}$ or $\{3\}$. See section 2.5 in this paper for more examples with $n = 5$.

For the inverted sum probabilities, we have $\Pr\big(n, \dot\Sigma(R) < 21 - \dot\sigma(n)\big) = 0$, $\Pr\big(n, \dot\Sigma(R) > 20\big) = 0$, and $\Pr\big(n, \dot\Sigma(R) = 19\big) = \Pr\big(n, \dot\Sigma(R) = 20\big) = \frac{1}{6^n}$ (inverted sums 19 and 20 have the same probability). In general, if all n dice in a roll have the same value, then $S = \{d\}$ for some $d \in D$ (and $|S| = 1$). If d equals 1, then $\dot\Sigma(R) = 20$. If d equals 2, then $\dot\Sigma(R) = 20$.

If the *multiplicity* of all elements in the multiset R equals 1, then $S = R$, and thus $\dot\Sigma(R) = 21 - \Sigma(R)$.[10] This is always the case when $n = 1$. Hence, inverted dice rolls with a single die are mathematically uninteresting. Nevertheless, let's have a look at this trivial situation, for completeness if nothing else.

2.1. Rolls with one die

When $n = 1$, $R \in \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$ and $\Pr\big(1, \dot\Sigma(R) = x\big) = \frac{1}{6}$ for all $x \in \{15, 16, 17, 18, 19, 20\}$. There is a one-to-one correspondence between the inverted sums and the regular sums, which doesn't give rise to any interesting mathematical questions (after all, sums with only a single addend are trivial). The equation $\dot\Sigma(R) = \Sigma(R)$ does not have any solutions since the intersection of the images of the functions $\dot\Sigma$ and Σ is empty.[11] Furthermore, $\dot\Sigma(R) + \Sigma(R)$ is always 21, so that is boring as well.

It would require some creative thinking to come up with any applicative situations at all for inverted sums of a single die. Maybe there are some fun games yet to be invented. For instance, players could take turns rolling a die with the goal of reaching 21 (or some other constant), both starting with 0 points. After each roll, the player chooses whether to use the shown value or the inverted value. If the regular value is chosen it is *added* to the player's points. If the inverted value is chosen, it is *subtracted* from the points (maybe only if the player has more points than what is required to win, otherwise added). The player who lands on 21 points using the lowest number of rolls wins the game. Modifications are needed, but this principle can be used in several types of games, including board games where a token is moved in a certain direction.

A single die has an even probability distribution. Since this distribution is not affected by subtracting the dice value from a constant (21 in this case), there is not really any reason to do so. Inverted dice rolls become much more interesting with multiple dice.

2.2. Rolls with two dice

Let us start with a table of the 36 possible outcomes of a roll with two dice. The rolls in the 15 red rows are duplicates of rolls displayed elsewhere in the table. Thus, the table contains 21 distinct rolls. The rolls in the 6 blue rows are special cases where $S \neq R$ (since $|S| = 1$ and $|R| = 2$), which means that $\dot\Sigma(R) \neq 21 - \Sigma(R)$.

R	$\Sigma(R)$	$\dot\Sigma(R)$	R	$\Sigma(R)$	$\dot\Sigma(R)$	R	$\Sigma(R)$	$\dot\Sigma(R)$
⚀ ⚀	2	20	⚂ ⚀	4	17	⚄ ⚀	6	15
⚀ ⚁	3	18	⚂ ⚁	5	16	⚄ ⚁	7	14
⚀ ⚂	4	17	⚂ ⚂	6	18	⚄ ⚂	8	13
⚀ ⚃	5	16	⚂ ⚃	7	14	⚄ ⚃	9	12
⚀ ⚄	6	15	⚂ ⚄	8	13	⚄ ⚄	10	16
⚀ ⚅	7	14	⚂ ⚅	9	12	⚄ ⚅	11	10
⚁ ⚀	3	18	⚃ ⚀	5	16	⚅ ⚀	7	14
⚁ ⚁	4	19	⚃ ⚁	6	15	⚅ ⚁	8	13
⚁ ⚂	5	16	⚃ ⚂	7	14	⚅ ⚂	9	12
⚁ ⚃	6	15	⚃ ⚃	8	17	⚅ ⚃	10	11
⚁ ⚄	7	14	⚃ ⚄	9	12	⚅ ⚄	11	10
⚁ ⚅	8	13	⚃ ⚅	10	11	⚅ ⚅	12	15

Table 3. The 36 possible outcomes of a two-dice roll with corresponding dice sums and inverted dice sums.

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[10] Since R is a *multiset* and S is a *set*, equality should be defined. Here, $S = R$ is equivalent to the existence of a *bijection* between the elements in S and R, such that $|S| = |R|$ and $\forall d(d \in S \iff d \in R)$.

[11] Solving the equation $\dot\Sigma(R) = \Sigma(R)$ means finding all distinct rolls R with the property that their regular sum is the same as their inverted sum. We'll get back to that problem in the following sections of this paper.

From Table 3, we can easily construct the probability tables for both the sum and the inverted sum (see Table 4). As mentioned earlier, $\dot{\sigma}(2) = \sigma(2)$, so the number of rows is the same (i.e., 11) for these probability tables. This makes the case $n = 2$ interesting.

The intersection of the images of the functions $\dot{\Sigma}$ and Σ is $\{10, 11, 12\}$, marked in blue below. The equation $\dot{\Sigma}(R) = \Sigma(R)$ does not have any solutions, since the distinct rolls with regular sums 10, 11, and 12, differs from the distinct rolls with inverted sums 10, 11, and 12, respectively, as seen in Table 3.

x	$\Pr(2, \Sigma(R) = x)$	x	$\Pr(2, \dot{\Sigma}(R) = x)$
2	1/36	10	2/36
3	2/36	11	2/36
4	3/36	12	4/36
5	4/36	13	4/36
6	5/36	14	6/36
7	6/36	15	5/36
8	5/36	16	5/36
9	4/36	17	3/36
10	3/36	18	3/36
11	2/36	19	1/36
12	1/36	20	1/36

Table 4. Possible values (x) for the sum and the inverted sum of a two-dice roll, with corresponding probabilities.

The table above shows us that the probability distribution for inverted dice sums differs from that of regular dice sums. The *two largest* inverted sums (19 and 20) are least likely to occur, and 14 is the inverted sum most likely to occur (with the same probability as the regular sum 7).

This can be used in several ways when it comes to dice games. For example, try playing Monopoly with inverted dice sums instead of regular dice sums (or add the possibility of players using inverted dice sums in certain situations). Inverted dice rolls with two dice can also be the foundation of *new* dice games, as discussed briefly in the previous section.

For $n = 2$, the probability of rolling the regular sum 11 and the inverted sum 11 is the same $(2/36)$. Table 4 shows that $x = 11$ is the only solution to $\Pr(2, \Sigma(R) = x) = \Pr(2, \dot{\Sigma}(R) = x)$. In fact, there are no other solutions to $\Pr(n, \Sigma(R) = x) = \Pr(n, \dot{\Sigma}(R) = x)$ for *all* n, except the trivial $x = 19$ for $n = 19$ and $x = 20$ for $n = 20$.

Lastly, it is worth mentioning that we can use the concept of inverted sums to define simple rules that allow us to interpret a roll of two dice in new ways. Below is an example of such rule (denoted Φ). It ranges from 3 to 20, i.e., it produces 18 sequential values instead of the 11 values we normally have for a sum of two dice. The rule Φ has an entertaining probability distribution, since both $\Pr(2, \Phi(R) = 5)$ and $\Pr(2, \Phi(R) = 7)$ are larger than $\Pr(2, \Phi(R) = 6)$. I leave it to the reader to construct the probability table for Φ.

Φ: If the two dice have different values and none of them is ⚄, use the sum. Otherwise, use the inverted sum.

The table below shows all 21 distinct rolls and their values according to the rule Φ defined above. The blue rows indicate that $\Phi(R) = \dot{\Sigma}(R)$. For all other rows, we have $\Phi(R) = \Sigma(R)$.

R	$\Phi(R)$	R	$\Phi(R)$	R	$\Phi(R)$
⚀⚀	20	⚁⚂	5	⚂⚅	9
⚀⚁	3	⚁⚃	6	⚃⚃	17
⚀⚂	4	⚁⚄	14	⚃⚄	12
⚀⚃	5	⚁⚅	8	⚃⚅	11
⚀⚄	15	⚂⚂	18	⚄⚄	16
⚀⚅	7	⚂⚃	7	⚄⚅	10
⚁⚁	19	⚂⚄	13	⚅⚅	15

Table 5. Distinct 2-dice rolls and their results according to the rule Φ defined above.

We can define all kinds of peculiar rules. The rule stated below always result in an *even number* between 2 and 18.

Ω: If the sum of the two dice is even, use the sum. Otherwise, use the inverted sum.

R	$\Omega(R)$	R	$\Omega(R)$	R	$\Omega(R)$
⚀⚀	2	⚁⚂	16	⚂⚅	12
⚀⚁	18	⚁⚃	6	⚃⚃	8
⚀⚂	4	⚁⚄	14	⚃⚄	12
⚀⚃	16	⚁⚅	8	⚃⚅	10
⚀⚄	6	⚂⚂	6	⚄⚄	10
⚀⚅	14	⚂⚃	14	⚄⚅	10
⚁⚁	4	⚂⚄	8	⚅⚅	12

Table 6. Distinct 2-dice rolls and their results according to the rule Ω defined above.

2.3. Rolls with three dice

With $n = 3$, the situation is more interesting. First of all, we have 16 possible regular sums (ranging from 3 to 18) but only 15 possible inverted sums (ranging from 6 to 20), so $\dot{\sigma}(3) \neq \sigma(3)$.

Secondly, among all possible rolls with 3 dice, we have a new kind of rolls, namely rolls where $1 < |S| < |R|$, i.e., rolls where some element in the multiset R have a multiplicity larger than 1 without all elements being the same, which is impossible for $n = 1$ and $n = 2$. Two examples are $R_1 =$ ⚀⚀⚁ and $R_2 =$ ⚀⚁⚁. We see that $\dot{\Sigma}(R_1) = \dot{\Sigma}(R_2)$, but $\Sigma(R_1) \neq \Sigma(R_2)$, and it is easy to see that for all such rolls, $\dot{\Sigma}(R) \neq 21 - \Sigma(R)$.

When $n = 3$ and $1 < |S| < |R|$, the set S corresponds to 2 distinct rolls (for $|S| = 1$ or $|S| = |R|$, the set S corresponds to 1 distinct roll). With three dice, we have 41 possible versions of the set S, all listed in the table below, together with the distinct rolls that give us each value of S and the inverted sum for these rolls, $\dot{\Sigma}(R)$.

S	R	$\dot{\Sigma}(R)$	S	R	$\dot{\Sigma}(R)$	S	R	$\dot{\Sigma}(R)$
⚀⚁⚂	⚀⚁⚂	15	⚀⚁	⚀⚀⚁ or ⚀⚁⚁	18	⚀	⚀⚀⚀	20
⚀⚁⚃	⚀⚁⚃	14	⚀⚂	⚀⚀⚂ or ⚀⚂⚂	17	⚁	⚁⚁⚁	19
⚀⚁⚄	⚀⚁⚄	13	⚀⚃	⚀⚀⚃ or ⚀⚃⚃	16	⚂	⚂⚂⚂	18
⚀⚁⚅	⚀⚁⚅	12	⚀⚄	⚀⚀⚄ or ⚀⚄⚄	15	⚃	⚃⚃⚃	17
⚀⚂⚃	⚀⚂⚃	13	⚀⚅	⚀⚀⚅ or ⚀⚅⚅	14	⚄	⚄⚄⚄	16
⚀⚂⚄	⚀⚂⚄	12	⚁⚂	⚁⚁⚂ or ⚁⚂⚂	16	⚅	⚅⚅⚅	15
⚀⚂⚅	⚀⚂⚅	11	⚁⚃	⚁⚁⚃ or ⚁⚃⚃	15			
⚀⚃⚄	⚀⚃⚄	11	⚁⚄	⚁⚁⚄ or ⚁⚄⚄	14			
⚀⚃⚅	⚀⚃⚅	10	⚁⚅	⚁⚁⚅ or ⚁⚅⚅	13			
⚀⚄⚅	⚀⚄⚅	9	⚂⚃	⚂⚂⚃ or ⚂⚃⚃	14			
⚁⚂⚃	⚁⚂⚃	12	⚂⚄	⚂⚂⚄ or ⚂⚄⚄	13			
⚁⚂⚄	⚁⚂⚄	11	⚂⚅	⚂⚂⚅ or ⚂⚅⚅	12			
⚁⚂⚅	⚁⚂⚅	9	⚃⚄	⚃⚃⚄ or ⚃⚄⚄	12			
⚁⚃⚄	⚁⚃⚄	10	⚃⚅	⚃⚃⚅ or ⚃⚅⚅	11			
⚁⚃⚅	⚁⚃⚅	9	⚄⚅	⚄⚄⚅ or ⚄⚅⚅	10			
⚁⚄⚅	⚁⚄⚅	8						
⚂⚃⚄	⚂⚃⚄	9						
⚂⚃⚅	⚂⚃⚅	8						
⚂⚄⚅	⚂⚄⚅	7						
⚃⚄⚅	⚃⚄⚅	6						

Table 7. Possible values of the set S for 3-dice rolls, the corresponding rolls, and $\dot{\Sigma}(R)$ for these rolls.

The rolls $R =$ ⚂⚂⚅ and $R =$ ⚂⚄⚄ marked in blue in the table above are solutions to the equation $\dot{\Sigma}(R) = \Sigma(R)$ mentioned earlier because the sum equals the inverted sum for these two rolls.[12]

Let's have a look at the probability table for rolls with three dice. The intersection of the images of the functions $\dot{\Sigma}$ and Σ is $\{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$, marked in blue below.

x	$\Pr(3, \Sigma(R) = x)$	x	$\Pr(3, \dot{\Sigma}(R) = x)$
3	1/216	6	6/216
4	3/216	7	6/216
5	6/216	8	12/216
6	10/216	9	18/216
7	15/216	10	24/216
8	21/216	11	24/216
9	25/216	12	30/216
10	27/216	13	24/216
11	27/216	14	24/216
12	25/216	15	19/216
13	21/216	16	13/216
14	15/216	17	7/216
15	10/216	18	7/216
16	6/216	19	1/216
17	3/216	20	1/216
18	1/216		

Table 8. Possible sums and inverted sums of a three-dice roll, with corresponding probabilities.

The table above shows us that the inverted sum most likely to occur is 12 (with a probability higher than that of any regular sum). As always with 6-sided dice, 19 and 20 are least likely to occur.

It is worth noticing that $\Pr(3, \dot{\Sigma}(R))$ has a near-symmetric nature around the value 12, since $\Pr\big(3, \dot{\Sigma}(R) = 12 - a\big)$ equals $\Pr\big(3, \dot{\Sigma}(R) = 12 + a\big)$ for $a \in \{1, 2\}$ and $\Pr\big(3, \dot{\Sigma}(R) = 12 + a\big) - 1$ for $a \in \{3, 4, 5, 6\}$.

When $n < 3$, for all x there are some y, such that $\Pr(n, \Sigma(R) = y) = \Pr\big(n, \dot{\Sigma}(R) = x\big)$. When $n = 3$, we see that $\Pr(3, \Sigma(R) = 3) = \Pr(3, \Sigma(R) = 18) = \Pr\big(3, \dot{\Sigma}(R) = 19\big) = \Pr\big(3, \dot{\Sigma}(R) = 20\big) = 1/216$, and $\Pr(3, \Sigma(R) = 5) = \Pr(3, \Sigma(R) = 16) = \Pr\big(3, \dot{\Sigma}(R) = 6\big) = \Pr\big(3, \dot{\Sigma}(R) = 7\big) = 6/216$. For $x \notin \{6, 7, 19, 20\}$, there are no values y, such that $\Pr(3, \Sigma(R) = y) = \Pr\big(3, \dot{\Sigma}(R) = x\big)$, with the impossible situations $y < 3$, $y > 18$, $x < 6$, and $x > 20$ as only exceptions.

It might be worth noticing that the probability of the regular sum being *odd* is $12/24$, while for the inverted sum we have $\Pr(3, \dot{\Sigma}(R)$ is *odd*$) = 11/24$. Also, $\Pr(3, \Sigma(R)$ is *prime*$) \approx 33.8\%$, while $\Pr(3, \dot{\Sigma}(R)$ is *prime*$) \approx 28.7\%$.

The concept of inverted sums can be used when creating new ways to interpret 3-dice rolls, as we did in the 2-dice case (the rules Φ and Ω in the previous section). As an example, let Δ be a rule stating that the result of a roll of three dice (R) is the difference between the regular and the inverted sum, i.e., $\Delta(R) = \big|\Sigma(R) - \dot{\Sigma}(R)\big|$. Then, the image of Δ is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 17\}$, which means $\Delta(R)$ can never be 11, 15 or 16.

Another example is the rule $\Sigma+\dot{\Sigma}$ defined to give us the sum of the regular sum and the inverted sum. The image of $\Sigma+\dot{\Sigma}$ is $\{21, 22, 23, 24, 25, 26, 27, 29, 31, 33\}$, which means $(\Sigma+\dot{\Sigma})(R)$ can never be 28, 30 or 32.

When applied on 3-dice rolls, the rule Ω (from the previous section) has the image $\{4, 6, 8, 10, 12, 13, 14, 15, 16, 17, 18, 20\}$. We see that it needs modification if we still want it only to give us *even* results.

It is an open problem to find a rule that always gives us a prime number. Table 7 is helpful when pursuing this.

[12] Since R is a multiset and multisets are unordered, multiple outcomes of a n-dice roll correspond to the same R, except for rolls on the form $R = \{d^n\}$, where d^n is shorthand for a single element $d \in D$ with the multiplicity n. For instance, the outcomes ⚄⚂⚄ and ⚄⚄⚂ are included in $R =$ ⚂⚄⚄ above. See section 2.5 for further discussion about this.

Using inverted sums of rolls with three dice can be useful in games. For instance, it is possible to play the game Inverted Dice™ with three dice (instead of five) by removing the top bonus section from the game. Maximum points, 295 (instead of 360), is much easier to achieve in this "light version" of the game. Try it!

2.4. Rolls with four dice

The case $n = 4$ is the first case where the lowest possible inverted sum (i.e., 3) is smaller than the lowest possible regular sum (i.e., 4), and the highest possible inverted sum (i.e., 20) is smaller than the highest possible regular sum (i.e., 24), see Table 2. We have $6^4 = 1296$ different outcomes of a roll with 4 dice. For each roll, R is one of 126 possible multisets (i.e., there are 126 distinct rolls), and S is one of 56 possible supporting sets for R.

Let us look at the probabilities. With four dice, the most likely inverted sum to occur is 9, as seen in the table below. The intersection of the images of the functions $\dot{\Sigma}$ and Σ is marked in blue, as in earlier tables.

x	$\Pr(4, \Sigma(R) = x)$	x	$\Pr(4, \dot{\Sigma}(R) = x)$
4	$1/1296 \approx 0.08\%$	3	$24/1296 \approx 1.85\%$
5	$4/1296 \approx 0.31\%$	4	$24/1296 \approx 1.85\%$
6	$10/1296 \approx 0.77\%$	5	$48/1296 \approx 3.70\%$
7	$20/1296 \approx 1.54\%$	6	$84/1296 \approx 6.48\%$
8	$35/1296 \approx 2.70\%$	7	$108/1296 \approx 8.33\%$
9	$56/1296 \approx 4.32\%$	8	$120/1296 \approx 9.26\%$
10	$80/1296 \approx 6.17\%$	9	$156/1296 \approx 12.04\%$
11	$104/1296 \approx 8.02\%$	10	$146/1296 \approx 11.27\%$
12	$125/1296 \approx 9.65\%$	11	$146/1296 \approx 11.27\%$
13	$140/1296 \approx 10.80\%$	12	$136/1296 \approx 10.49\%$
14	$146/1296 \approx 11.27\%$	13	$100/1296 \approx 7.72\%$
15	$140/1296 \approx 10.80\%$	14	$78/1296 \approx 6.02\%$
16	$125/1296 \approx 9.65\%$	15	$65/1296 \approx 5.02\%$
17	$104/1296 \approx 8.02\%$	16	$29/1296 \approx 2.24\%$
18	$80/1296 \approx 6.17\%$	17	$15/1296 \approx 1.16\%$
19	$56/1296 \approx 4.32\%$	18	$15/1296 \approx 1.16\%$
20	$35/1296 \approx 2.70\%$	19	$1/1296 \approx 0.08\%$
21	$20/1296 \approx 1.54\%$	20	$1/1296 \approx 0.08\%$
22	$10/1296 \approx 0.77\%$		
23	$4/1296 \approx 0.31\%$		
24	$1/1296 \approx 0.08\%$		

Table 9. Possible regular and inverted sums of a four-dice roll, with corresponding probabilities.

The table shows that $\Pr(4, \Sigma(R) = 14) = \Pr\big(4, \dot{\Sigma}(R) = 10\big) = \Pr\big(4, \dot{\Sigma}(R) = 11\big)$, while for $y \neq 14$, there are no x, such that $\Pr(4, \Sigma(R) = y) = \Pr\big(4, \dot{\Sigma}(R) = x\big)$.

It's a fun fact that the probability of getting a *regular* sum that is a prime number is exactly $1/3$, and the probability of the regular sum being odd is exactly $1/2$. For the inverted sum, $\Pr(4, \dot{\Sigma}(R)$ is *odd*$) \approx 51.1\%$ and $\Pr(4, \dot{\Sigma}(R)$ is *prime*$) \approx 34.1\%$, which is not very exciting.

Of more interest is the fact that the equation $\dot{\Sigma}(R) = \Sigma(R)$ has 8 solutions for $n = 4$: $R \in \{\{1,2,5^2\}, \{1^2,3,6\}, \{1^2,4,5\}, \{2,3^2,4\}, \{2^2,5^2\}, \{1,3^2,5\}, \{1^2,6^2\}, \{3^2,4^2\}\}$, where the upper indices represent the multiplicities greater than 1. In the next section, we'll see that there are fewer solutions when $n = 5$.

2.5. Rolls with five dice

This case is perhaps the most beautiful, because with 5 dice we have exactly 20 possible inverted sums, and they range from 1 to 20. In other words, five 6-sided dice can together form *one* 20-sided die (with an unusual probability distribution). This is the foundation of the game Inverted Dice™. Other games are yet to be invented.

So far, we have been using the concept of *distinct rolls* a bit loosely, and the set S also needs further clarification. Let us define these terms properly before we continue our presentation of the case $n = 5$.

An *outcome* of a roll of n dice is a *sequence* of length n, with elements (dice values) $d_k \in D$, for $k = 1, 2, \dots, n$.[13] Let $\mathcal{O}_{6,n}$ be the set of all possible outcomes of a roll with n dice (6-faced). Then, $|\mathcal{O}_{6,n}| = 6^n$. It is easy to see, that $\mathcal{O}_{6,n}$ is also the set of all possible outcomes of n rolls of *one* die. If we roll *one* die *five* times, we get a sequence $(d_1, d_2, d_3, d_4, d_5)$. Since the five values d_i are generated separately, it is easy to keep track of them. But if you roll *five* dice *one* time (in short, *a roll of 5 dice* or *a 5-dice roll*), there is no way for us to distinguish outcomes such as ⚀⚀⚀⚀⚅ and ⚀⚀⚅⚀⚀. Hence, we need the concept of *distinct* rolls.

Let $\mathcal{R}_{6,n}$ be the set of all possible multisets of cardinality n with elements taken from the 6 dice values in D. Then each element in $\mathcal{R}_{6,n}$ is called a *distinct roll* of n dice (6-faced). Each such roll $R \in \mathcal{R}_{6,n}$ is given by $R = \{1^{m_1}, 2^{m_2}, \dots, 6^{m_6}\}$, where $0 \le m_k \le n$ (for $k = 1, 2, \dots, 6$) are the multiplicities, whose sum is n.[14] For each outcome $O \in \mathcal{O}_{6,n}$, there is exactly one distinct roll $R \in \mathcal{R}_{6,n}$ such that all elements d_k in R with multiplicity m_k appear as terms in O exactly m_k times.[15] Thus, R can be defined as a function $R\colon \mathcal{O}_{6,n} \longrightarrow \mathcal{R}_{6,n}$. There are $|\mathcal{O}_{6,5}| = 6^5 = 7776$ different outcomes of a roll with 5 dice, but only $|\mathcal{R}_{6,5}| = 252$ distinct rolls.[16]

We have 62 possible sets S, all shown in the table below (sorted by $\dot{\Sigma}(S)$ in ascending order). For each S, the table also contains the inverted sum and the number of distinct 5-dice rolls (multisets with cardinality 5) for which S is the support. That number is denoted r_S below. In other words, $r_S = |\{R \in \mathcal{R}_{6,5} : \mathrm{Supp}(R) = S\}|$.

S	r_S	$\dot{\Sigma}(S)$	S	r_S	$\dot{\Sigma}(S)$	S	r_S	$\dot{\Sigma}(S)$
⚁⚂⚃⚄⚅	1	1	⚀⚁⚂⚅	4	9	⚀⚂⚃	6	13
⚀⚂⚃⚄⚅	1	2	⚀⚁⚃⚄	4	9	⚁⚅	4	13
⚀⚁⚃⚄⚅	1	3	⚀⚄⚅	6	9	⚂⚄	4	13
⚂⚃⚄⚅	4	3	⚁⚃⚅	6	9	⚀⚁⚃	6	14
⚀⚁⚂⚄⚅	1	4	⚂⚃⚄	6	9	⚀⚅	4	14
⚁⚃⚄⚅	4	4	⚀⚁⚂⚄	4	10	⚁⚄	4	14
⚀⚁⚂⚃⚅	1	5	⚀⚃⚅	6	10	⚂⚃	4	14
⚀⚃⚄⚅	4	5	⚁⚂⚅	6	10	⚀⚁⚂	6	15
⚁⚂⚄⚅	4	5	⚁⚃⚄	6	10	⚀⚄	4	15
⚀⚁⚂⚃⚄	1	6	⚄⚅	4	10	⚁⚃	4	15
⚀⚂⚄⚅	4	6	⚀⚁⚂⚃	4	11	⚅	1	15
⚁⚂⚃⚅	4	6	⚀⚂⚅	6	11	⚀⚃	4	16
⚃⚄⚅	6	6	⚀⚃⚄	6	11	⚁⚂	4	16
⚀⚁⚄⚅	4	7	⚁⚂⚄	6	11	⚄	1	16
⚀⚂⚃⚅	4	7	⚃⚅	4	11	⚀⚂	4	17
⚁⚂⚃⚄	4	7	⚀⚁⚅	6	12	⚃	1	17
⚂⚄⚅	6	7	⚀⚂⚄	6	12	⚀⚁	4	18
⚀⚁⚃⚅	4	8	⚁⚂⚃	6	12	⚂	1	18
⚀⚂⚃⚄	4	8	⚂⚅	4	12	⚁	1	19
⚁⚄⚅	6	8	⚃⚄	4	12	⚀	1	20
⚂⚃⚅	6	8	⚀⚁⚄	6	13		252	

Table 10. Possible values of the set S for 5-dice rolls, the number of corresponding rolls, and $\dot{\Sigma}(R)$ for these rolls.

An example: With $S =$ ⚂⚃⚄⚅, we have $r_S = 4$ (see Table 10), since $S = \mathrm{Supp}($⚂⚂⚃⚄⚅$)$, $S = \mathrm{Supp}($⚂⚃⚃⚄⚅$)$, $S = \mathrm{Supp}($⚂⚃⚄⚄⚅$)$, and $S = \mathrm{Supp}($⚂⚃⚄⚅⚅$)$.

[13] Sequences can be regarded as *ordered* multisets. See https://en.wikipedia.org/wiki/Sequence for a discussion about the concept.

[14] Here, a value d_k having a multiplicity of *zero* means that $d_k \notin R$. For instance, $R = \{1^0, 2^0, 3^1, 4^0, 5^2, 6^0\} =$ ⚂⚄⚄ $(n = 3)$.

[15] If we regard outcomes as ordered multisets, $\mathcal{R}_{6,n}$ is a *family of subsets* of $\mathcal{O}_{6,n}$, see https://en.wikipedia.org/wiki/Family_of_sets.

[16] The number 252 is the *multiset coefficient*, given by the binomial coefficient $\binom{f+n-1}{n}$, i.e., the number of multisets of cardinality n, with elements taken from a finite set of cardinality f. In this case, $n = 5$ (five dice) and $f = |D| = 6$ (six-faced dice).

In Table 10, we see that $r_S = 1, 4, 6, 4, 1$ for $|S| = 1, 2, 3, 4, 5$, respectively. The *fourth* row in Pascal's triangle is also 1, 4, 6, 4, 1 (the top row is the *0th*). We'll get back to these binomial coefficients in section 2.7.

We have previously defined the set S by converting each corresponding roll R from a multiset to a set, thus removing duplicate elements from each R. In other words, for each roll R we have $S = R \cap D$.[17] Defining S this way is useful when we need to find the inverted sum of a roll, for instance when playing games. To be exact, with this approach, S is a function, $S: \mathcal{R}_{6,n} \longrightarrow \mathcal{S}_{6,n}$, where the codomain $\mathcal{S}_{6,n}$ is the set of all possible sets S for n 6-faced dice.

The cardinality of the domain $\mathcal{R}_{6,n}$ increases as n increases, while for $n > 5$, the cardinality of $\mathcal{S}_{6,n}$ is always 63 (for $n = 5$, it is 62, as mentioned before). To see this, let $\mathcal{P}(D)$ be the *power set* of the set D, i.e., $\mathcal{P}(D)$ is the set of all subsets of D, including both D itself and $\emptyset$.[18] We have $|\mathcal{P}(D)| = 2^6 = 64$, so $|\mathcal{S}_{6,5}| = 64 - 2 = 62$, since neither D nor $\emptyset$ is in $\mathcal{S}_{6,5}$.

For each inverted sum x (where $0 \leq x \leq 20$), let s_x be the number of sets S with the inverted sum x. In other words, $s_x = |\{S \epsilon \mathcal{S}_{6,n}: \dot{\Sigma}(S) = x\}|$. Also, let $r_x = |\{R \in \mathcal{R}_{6,n}: \dot{\Sigma}(R) = x\}|$ and $o_x = |\{O \in \mathcal{O}_{6,n}: \dot{\Sigma}(O) = x\}|$.[19] Now, let's look at the probabilities for 5-dice rolls.

x	s_x	r_x	o_x	$\Pr(5, \dot{\Sigma}(R) = x)$
1	1	1	120	$120/7776 \approx 1.543\%$
2	1	1	120	$120/7776 \approx 1.543\%$
3	2	5	360	$360/7776 \approx 4.630\%$
4	2	5	360	$360/7776 \approx 4.630\%$
5	3	9	600	$600/7776 \approx 7.716\%$
6	4	15	750	$750/7776 \approx 9.645\%$
7	4	18	870	$870/7776 \approx 11.188\%$
8	4	20	780	$780/7776 \approx 10.031\%$
9	5	26	930	$930/7776 \approx 11.960\%$
10	5	26	720	$720/7776 \approx 9.259\%$
11	5	26	720	$720/7776 \approx 9.259\%$
12	5	26	510	$510/7776 \approx 6.559\%$
13	4	20	360	$360/7776 \approx 4.630\%$
14	4	18	240	$240/7776 \approx 3.086\%$
15	4	15	211	$211/7776 \approx 2.713\%$
16	3	9	61	$61/7776 \approx 0.784\%$
17	2	5	31	$31/7776 \approx 0.399\%$
18	2	5	31	$31/7776 \approx 0.399\%$
19	1	1	1	$1/7776 \approx 0.013\%$
20	1	1	1	$1/7776 \approx 0.013\%$
	62	252	7776	

Table 11. Possible inverted sums (x) of a five-dice roll, with the corresponding number of sets S (s_x), the corresponding number of distinct rolls (r_x), and the corresponding number of outcomes (o_x) for each x.

The probability distribution is quite strange, as seen in Table 11. Each of the inverted sums 9 to 12, corresponds to 26 distinct rolls (and 5 different sets S). The inverted sum most likely to occur is $x = 9$, followed by $x = 7$ (corresponding to only 18 distinct rolls), $x = 8$ (corresponding to 20 distinct rolls), and $x = 6$ (15 distinct rolls). The relationship between s_x, r_x and o_x is not intuitively clear, as seen in Table 11.

[17] To my knowledge, there is no standard notation for converting a multiset to a set. Hence, using the intersection as a *set constructing operator* is probably one of the least confusing ways to notate such conversions. I have seen several other suggestions, such as defining the brackets to be set constructing operators, so $S = \{x: x \in R\}$ or $S = \bigcup_{x \in R}\{x\}$, but I use the brackets for multisets as well, which makes this a poor alternative here. We could introduce a *set constructor*, Set, and write $S = \text{Set}(R)$, but it seems unnecessary.

[18] For more information about power sets, see https://en.wikipedia.org/wiki/Power_set.

[19] Here, $\dot{\Sigma}(O)$ is the inverted sum of the *sequence* O. Strictly speaking, we have not defined what this means, so let's define it as $\dot{\Sigma}(R(O))$, where $R(O)$ is the value of R applied to the argument O (with R being the function $R: \mathcal{O}_{6,n} \longrightarrow \mathcal{R}_{6,n}$ in this context). As earlier, $\dot{\Sigma}(R)$ and $\dot{\Sigma}(S)$ are the inverted sums of the *multiset* R and the *set* S, respectively.

The intersection of the images of the functions $\dot{\Sigma}$ and Σ is the integers from 5 to 20, and the union is $\{1, 2, \ldots, 30\}$. It is left to the reader to find a *simple* rule (like the ones in previous sections) that allows us to interpret a roll of 5 dice as a value between 1 and 30. Such rule could be advantageous in games.

We have $\Pr(5, \Sigma(R) = 17) = \Pr(5, \Sigma(R) = 18) = \Pr\big(5, \dot{\Sigma}(R) = 8\big) = 780/7776$, while for $x \neq 8$, there is no y, such that $\Pr(4, \Sigma(R) = y) = \Pr\big(4, \dot{\Sigma}(R) = x\big)$, except the trivial solutions $y < 5$, $y > 30$, $x < 1$, and $x > 20$.

The equation $\dot{\Sigma}(R) = \Sigma(R)$ has 3 solutions for $n = 5$: $R \in \{\{1^2, 3, 4^2\}, \{1^2, 2^2, 6\}, \{1^2, 2, 3, 4\}\}$.[20]

We have $\Pr(5, \dot{\Sigma}(R)$ is *odd*$) \approx 54.1\%$ and $\Pr(5, \dot{\Sigma}(R)$ is *prime*$) \approx 39.4\%$, while for regular sums, we have $\Pr(5, \Sigma(R)$ is *odd*$) = 50\%$ and $\Pr(5, \Sigma(R)$ is *prime*$) \approx 31.7\%$.

The image of $\Sigma + \dot{\Sigma}$ (sum of regular and inverted sum, as earlier) is $\{21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 45\}$. Not much fun there.

Now, let $\Delta(R) = \big|\Sigma(R) - \dot{\Sigma}(R)\big|$, as before. Then, the image of Δ contains all 22 integers from 0 to 21, and the probability distribution for Δ is interesting, because of its dispersion. It has a low variance (4.11) compared to the variance of the probability distribution of both regular (13.17) and inverted (16.14) sums.[21] Below is the probability table.

x	$\Pr(5, \Delta(R) = x)$	x	$\Pr(5, \Delta(R) = x)$	x	$\Pr(5, \Delta(R) = x)$
0	$120/7776 \approx 1.54\%$	8	$390/7776 \approx 5.02\%$	16	$185/7776 \approx 2.38\%$
1	$405/7776 \approx 5.21\%$	9	$682/7776 \approx 8.77\%$	17	$395/7776 \approx 5.08\%$
2	$320/7776 \approx 4.12\%$	10	$390/7776 \approx 5.02\%$	18	$190/7776 \approx 2.44\%$
3	$422/7776 \approx 5.43\%$	11	$580/7776 \approx 7.46\%$	19	$315/7776 \approx 4.05\%$
4	$395/7776 \approx 5.08\%$	12	$410/7776 \approx 5.27\%$	20	$90/7776 \approx 1.16\%$
5	$460/7776 \approx 5.92\%$	13	$555/7776 \approx 7.14\%$	21	$80/7776 \approx 1.03\%$
6	$435/7776 \approx 5.59\%$	14	$180/7776 \approx 2.31\%$		
7	$280/7776 \approx 3.60\%$	15	$497/7776 \approx 6.39\%$		

Table 12. Possible values of $\Delta(R) = \big|\Sigma(R) - \dot{\Sigma}(R)\big|$ for a five-dice roll, with corresponding probabilities.

The rule Δ defined above might be useful when creating new games. The highest values (20 and 21) are hard to get in one roll, but within reach in a normal dice game.

That being said, $\Pr\big(5, \dot{\Sigma}(R) = 19\big) = \Pr\big(5, \dot{\Sigma}(R) = 20\big) = \frac{1}{7776}$ is what makes it almost impossible to get maximum points in Inverted Dice™, where you have *three* rolls every turn. And that's part of the game.

2.6. Rolls with six dice

The case with 6 dice is also interesting, since it is the first case which has 21 different possible inverted sums, and the first case where it is possible to get the inverted sum *zero*. Maybe it is also the last case that has some *practical* use, since rolling more than 6 physical dice at the same time is difficult (most dice cups are of limited size). Here's the probability table:

x	$\Pr(6, \dot{\Sigma}(R) = x)$	x	$\Pr(6, \dot{\Sigma}(R) = x)$	x	$\Pr(6, \dot{\Sigma}(R) = x)$
0	$720/46656 \approx 1.5432\%$	7	$5220/46656 \approx 11.1883\%$	14	$726/46656 \approx 1.5561\%$
1	$1800/46656 \approx 3.8580\%$	8	$4200/46656 \approx 9.0021\%$	15	$665/46656 \approx 1.4253\%$
2	$1800/46656 \approx 3.8580\%$	9	$4740/46656 \approx 10.1595\%$	16	$125/46656 \approx 0.2679\%$
3	$3360/46656 \approx 7.2016\%$	10	$3242/46656 \approx 6.9487\%$	17	$63/46656 \approx 0.1350\%$
4	$3360/46656 \approx 7.2016\%$	11	$3242/46656 \approx 6.9487\%$	18	$63/46656 \approx 0.1350\%$
5	$4920/46656 \approx 10.5453\%$	12	$1744/46656 \approx 3.7380\%$	19	$1/46656 \approx 0.0021\%$
6	$5460/46656 \approx 11.7027\%$	13	$1204/46656 \approx 2.5806\%$	20	$1/46656 \approx 0.0021\%$

Table 13. Possible values of $\dot{\Sigma}(R)$ for a five-dice roll, with corresponding probabilities.

The inverted sum that is most likely to occur is 6, followed by 7, 5, 9, 8 (in that order) as seen in the table above.

[20] The upper indices represent the multiplicities greater than 1 (as in section 2.4).

[21] *Variance* might not be the most adequate measure of dispersion here, but supplemented with Table 12, it illustrates the situation sufficiently.

We have $|\mathcal{O}_{6,6}| = 6^6 = 46656$ different outcomes of a 6-dice roll, and there are $|\mathcal{R}_{6,6}| = 462$ distinct rolls and $|\mathcal{S}_{6,6}| = 63$ different sets S. There are 8 distinct rolls R satisfying $\dot{\Sigma}(R) = \Sigma(R)$:

⚀⚀⚀⚀⚁⚅ (12) ⚀⚀⚀⚀⚂⚄ (12) ⚀⚀⚀⚂⚂⚃ (13) ⚀⚀⚁⚁⚁⚄ (13)
⚀⚀⚁⚁⚃⚃ (14) ⚀⚁⚂⚂⚂⚂ (15) ⚁⚁⚂⚂⚂⚂ (16) ⚂⚂⚂⚂⚂⚂ (18)

The rule Δ (i.e., $\Delta(R) = |\Sigma(R) - \dot{\Sigma}(R)|$, as before) produces all 28 integers from 0 to 27. I leave it to the reader to investigate the probabilities for Δ and other rules, as discussed above.

The case $n = 6$ might be useful in games, even though the chance of rolling the inverted sums 19 and 20 are *very* low. When constructing games, these two rolls could be given an extraordinary meaning (such as *doubling all points*, immediately *ending the game*, giving the player *an extra roll,* etc.). We could also let the inverted sum 0 have a special meaning in the game, like with a roulette. Only our creativity sets the limit.

There is more to be said here, but I leave it to the reader to further explore the world of inverted dice rolls with *six* standard dice. In the next section, we will summarize what we know about rolls of n dice, and take a closer look at their probabilities, before briefly discussing non-standard dice in section 3.

2.7. Rolls with n dice

As before, let $D = \{1, 2, 3, 4, 5, 6\}$ be the set of possible dice values, let $\mathcal{O}_{6,n}$ be the set of possible outcomes of a roll with n dice, and let $\mathcal{R}_{6,n}$ be the set of all distinct n-dice rolls (multisets of cardinality n). Elements in $\mathcal{O}_{6,n}$ are sequences with n terms, and elements in $\mathcal{R}_{6,n}$ are multisets of cardinality n. We have $|\mathcal{O}_{6,n}| = 6^n$ and $|\mathcal{R}_{6,n}| = \binom{6+n-1}{n} = \frac{(n+5)!}{120 \cdot n!}$.

Let $\mathcal{S}_{6,n} = \{S \in \mathcal{P}^+(D): |S| \leq n\}$, where $\mathcal{P}^+(D)$ is the set of non-empty subsets of D, i.e., $\mathcal{P}^+(D) = \mathcal{P}(D) \setminus \emptyset$.[22] In other words, $\mathcal{S}_{6,n}$ is the subset of $\mathcal{P}(D)$ having elements of cardinality n or lower (but non-zero). For $n \geq |D|$, $\mathcal{S}_{6,n} = \mathcal{P}^+(D)$, as seen in previous sections. The cardinality of $\mathcal{S}_{6,n}$ is given by

$$|\mathcal{S}_{6,n}| = \sum_{k=1}^{n} \binom{6}{k}$$

where the terms $\binom{6}{k}$ equals 0 for $k > 6$. Thus, $|\mathcal{S}_{6,n}| = 6, 21, 41, 56, 62$ for $n = 1, 2, 3, 4, 5$, as seen in the previous discussion. For $n > 5$, $|\mathcal{S}_{6,n}|$ equals 63.

We have previously referred to the *image of* Σ and the *image of* $\dot{\Sigma}$, so let us define the functions Σ and $\dot{\Sigma}$ properly. Let $\Sigma: \mathcal{R}_{6,n} \longrightarrow \mathbb{N}$ be a function such that

$$\Sigma(R) = \Sigma(\{d_k^{m_k}\}) = \sum_{d_k \in D} d_k \cdot m_k = \sum_{k=1}^{6} k \cdot m_k$$

where $R \in \mathcal{R}_{6,n}$ is a multiset $\{d_k^{m_k}\}$ of cardinality n.[23] As before, the upper indices m_k represent the multiplicities, with $m_k = 0$ meaning $d_k \notin R$. In short, $\Sigma(R)$ is the sum of all n dice values in the roll R, with repetitions.

Let $\Sigma[\mathcal{R}_{6,n}] = \{\Sigma(R): R \in \mathcal{R}_{6,n}\}$ denote *the image of* $\mathcal{R}_{6,n}$ *under* Σ, in short (as earlier), *the image of* Σ. It is easy to see that $\Sigma[\mathcal{R}_{6,n}] = \{n, n+1, n+2, \ldots, 6n\}$ and $\sigma(n) = |\Sigma[\mathcal{R}_{6,n}]| = 5n + 1$ (see Table 2).

Let $\dot{\Sigma}: \mathcal{R}_{6,n} \longrightarrow \mathbb{N}$ be a function such that

$$\dot{\Sigma}(R) = 21 - \sum_{d \in S} d$$

where, as earlier, $S = R \cap D$. In short, $\dot{\Sigma}(R)$ is the sum of all n dice values *not* in the roll R, i.e., the *inverted sum*.

Let $\dot{\Sigma}[\mathcal{R}_{6,n}] = \{\dot{\Sigma}(R): R \in \mathcal{R}_{6,n}\}$ denote *the image of* $\mathcal{R}_{6,n}$ *under* $\dot{\Sigma}$, in short, *the image of* $\dot{\Sigma}$. For $n > 5$, we have $\dot{\Sigma}[\mathcal{R}_{6,n}] = \{0, 1, 2, \ldots, 20\}$ and $\dot{\sigma}(n) = |\dot{\Sigma}[\mathcal{R}_{6,n}]| = 21$. For $n \leq 5$, I refer to Table 2.

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[22] As in section 2.5, $\mathcal{P}(D)$ is the *power set* of D.

[23] To the best of my knowledge, there is no standard notation for summation of elements in a multiset, but I hope the notation used here is relatively clear.

The table below contains the intersection $\Sigma[\mathcal{R}_{6,n}] \cap \dot{\Sigma}[\mathcal{R}_{6,n}]$ and the union $\Sigma[\mathcal{R}_{6,n}] \cup \dot{\Sigma}[\mathcal{R}_{6,n}]$ for all n.

n	$\Sigma[\mathcal{R}_{6,n}] \cap \dot{\Sigma}[\mathcal{R}_{6,n}]$	$\Sigma[\mathcal{R}_{6,n}] \cup \dot{\Sigma}[\mathcal{R}_{6,n}]$	n	$\Sigma[\mathcal{R}_{6,n}] \cap \dot{\Sigma}[\mathcal{R}_{6,n}]$	$\Sigma[\mathcal{R}_{6,n}] \cup \dot{\Sigma}[\mathcal{R}_{6,n}]$
1	$\emptyset$	$D \cup \{15, 16, \dots, 20\}$	12	$\{12, 13, \dots, 20\}$	$\{0, 1, \dots, 72\}$
2	$\{10, 11, 12\}$	$\{2, 3, \dots, 20\}$	13	$\{13, 14, \dots, 20\}$	$\{0, 1, \dots, 78\}$
3	$\{6, 7, \dots, 18\}$	$\{3, 4, \dots, 20\}$	14	$\{14, 15, \dots, 20\}$	$\{0, 1, \dots, 84\}$
4	$\{4, 5, \dots, 20\}$	$\{3, 4, \dots, 24\}$	15	$\{15, 16, \dots, 20\}$	$\{0, 1, \dots, 90\}$
5	$\{5, 6, \dots, 20\}$	$\{1, 2, \dots, 30\}$	16	$\{16, 17, \dots, 20\}$	$\{0, 1, \dots, 96\}$
6	$\{6, 7, \dots, 20\}$	$\{0, 1, \dots, 36\}$	17	$\{17, 18, 19, 20\}$	$\{0, 1, \dots, 102\}$
7	$\{7, 8, \dots, 20\}$	$\{0, 1, \dots, 42\}$	18	$\{18, 19, 20\}$	$\{0, 1, \dots, 108\}$
8	$\{8, 9, \dots, 20\}$	$\{0, 1, \dots, 48\}$	19	$\{19, 20\}$	$\{0, 1, \dots, 114\}$
9	$\{9, 10, \dots, 20\}$	$\{0, 1, \dots, 54\}$	20	$\{20\}$	$\{0, 1, \dots, 120\}$
10	$\{10, 11, \dots, 20\}$	$\{0, 1, \dots, 60\}$	21	$\emptyset$	$\{0, 1, \dots, 126\}$
11	$\{11, 12, \dots, 20\}$	$\{0, 1, \dots, 66\}$	$\boldsymbol{n > 21}$	$\emptyset$	$\{0, 1, \dots, 20\} \cup \{n, n+1, \dots, 6n\}$

Table 14. Intersections and unions of the images $\Sigma[\mathcal{R}_{6,n}]$ and $\dot{\Sigma}[\mathcal{R}_{6,n}]$.

There are some interesting things to be learned from Table 14. For $n = 1$ or $n \geq 22$, there exist some numbers x between the minimum and the maximum of $\Sigma[\mathcal{R}_{6,n}] \cup \dot{\Sigma}[\mathcal{R}_{6,n}]$ such that x is not itself an element of the union, while for $2 \leq n \leq 21$, the union of can be regarded as an integer sequence without such "holes". Also, it is worth noticing, that for $6 \leq n \leq 21$, these sequences have the form 0, 1, …, $6n$, i.e., they have $6n + 1$ terms. This might be relevant when making new games since it allows us to interpret for instance a roll of six dice (with a total of 36 faces) as one of 37 different values, in a relatively simple way.

When it comes to the intersection of $\Sigma[\mathcal{R}_{6,n}]$ and $\dot{\Sigma}[\mathcal{R}_{6,n}]$, Table 14 shows us that for $4 \leq n \leq 20$, it is simply the numbers from n to 20. Hence, $\Sigma[\mathcal{R}_{6,4}] \cap \dot{\Sigma}[\mathcal{R}_{6,4}]$ has the highest cardinality among all such intersections.

In previous sections, we have listed all rolls R satisfying the equation $\Sigma(R) = \dot{\Sigma}(R)$ for $n < 7$. Table 14 shows us that no solutions can exist for $n > 20$ (since $\Sigma[\mathcal{R}_{6,n}] \cap \dot{\Sigma}[\mathcal{R}_{6,n}]$ is empty when $n > 20$). Interestingly, there are rather few solutions even in cases with a large number of possible outcomes. For instance, with $n = 13$, we have $|\mathcal{O}_{6,13}| = 13060694016$ possible outcomes, but only 3 solutions, $R = \{1^{12}, 4\}$, $R = \{1^{11}, 3^2\}$, and $R = \{1^8, 2^5\}$. It is left to the reader to find the rest of the solutions in the cases $n \geq 7$.

The probability of getting the inverted sum 0 in an n-dice roll approaches 100% as n approaches infinity, and we do not need hundreds of dice to see this in the probability distribution. As an example, $Pr(15, \dot{\Sigma}(R) = 0)$ is approximately 64.4% and $Pr(20, \dot{\Sigma}(R) = 0)$ is approximately 84.8%.

The sets $\mathcal{O}_{6,n}$ and $\mathcal{R}_{6,n}$ depend on n. But for $n > 5$, the set $\mathcal{S}_{6,n}$ is a constant set containing 63 elements (being sets themselves), even though the probability of each $S \in \mathcal{S}_{6,n}$ occurring depends on n. Let us recall our definitions from section 2.5. For each $S \in \mathcal{S}_{6,n}$, let $r_S = |\{R \in \mathcal{R}_{6,n} : \mathrm{Supp}(R) = S\}|$. For each inverted dice sum x, let $s_x = |\{S \epsilon \mathcal{S}_{6,n} : \dot{\Sigma}(S) = x\}|$, $r_x = |\{R \in \mathcal{R}_{6,n} : \dot{\Sigma}(R) = x\}|$, and $o_x = |\{O \in \mathcal{O}_{6,n} : \dot{\Sigma}(O) = x\}|$.

We can calculate each probability as $\Pr(n, \dot{\Sigma}(R) = x) = \frac{o_x}{6^n}$, that part is easy. The challenge is to find o_x (to count the outcomes) for each x. Table 10 in section 2.5 contains all S for $n = 5$. Adding $S =$ ⚀⚁⚂⚃⚄⚅ to that table gives us a complete list of all 63 sets in $\mathcal{S}_{6,n}$ for $n > 5$. For each inverted sum x, Table 15 below shows s_x and the cardinalities $|S|$ for each of the different sets S corresponding to x (in the sense $\dot{\Sigma}(S) = x$). For each x, let $\mathcal{C}_x$ denote the multiset consisting of these cardinalities. Then obviously, we have $s_x = |\mathcal{C}_x|$.

x	s_x	$\mathcal{C}_x$	x	s_x	$\mathcal{C}_x$	x	s_x	$\mathcal{C}_x$	x	s_x	$\mathcal{C}_x$
0	1	{6}	6	4	{3, 4, 4, 5}	12	5	{2, 2, 3, 3, 3}	18	2	{1, 2}
1	1	{5}	7	4	{3, 4, 4, 4}	13	4	{2, 2, 3, 3}	19	1	{1}
2	1	{5}	8	4	{3, 3, 4, 4}	14	4	{2, 2, 2, 3}	20	1	{1}
3	2	{4, 5}	9	5	{3, 3, 3, 4, 4}	15	4	{1, 2, 2, 3}		63	
4	2	{4, 5}	10	5	{2, 3, 3, 3, 4}	16	3	{1, 2, 2}			
5	3	{4, 4, 5}	11	5	{2, 3, 3, 3, 4}	17	2	{1, 2}			

Table 15. Number of sets S (s_x) and the cardinalities for each S ($\mathcal{C}_x$) for each inverted sum x.

Before, we proceed, an example might be necessary. For $x = 6$, we have $s_6 = 4$ since there are 4 different sets S, such that $\dot{\Sigma}(S) = 6$. These sets are {4, 5, 6}, {1, 3, 5, 6}, {2, 3, 4, 6}, and {1, 2, 3, 4, 5}, and their cardinalities are 3, 4, 4, 5, respectively, so $\mathcal{C}_6 = \{3, 4, 4, 5\}$.

The 21 multisets $\mathcal{C}_x$ do not depend on n, only on the values in D. Table 15 shows us, that $\mathcal{C}_1 = \mathcal{C}_2$, $\mathcal{C}_3 = \mathcal{C}_4$, $\mathcal{C}_{10} = \mathcal{C}_{11}$, $\mathcal{C}_{17} = \mathcal{C}_{18}$, and $\mathcal{C}_{19} = \mathcal{C}_{20}$. As we shall see below, this implies that $o_1 = o_2$, $o_3 = o_4$, $o_{10} = o_{11}$, $o_{17} = o_{18}$, and $o_{19} = o_{20}$, so for *all* n, $\Pr(n, \dot{\Sigma}(R) = x) = \Pr(n, \dot{\Sigma}(R) = x + 1)$ when $x \in \{1, 3, 10, 17, 19\}$. After all, there *is* some symmetrical beauty hidden in the probabilities of inverted dice sums.

The number of distinct rolls corresponding to each S is given by $r_S = \binom{n-1}{|S|-1}$, where $|S|$ is cardinality of each S. This gives us

$$r_x = \sum_{c \in \mathcal{C}_x} \binom{n-1}{c-1} = \sum_{c \in \mathcal{C}_x} \frac{\Gamma(n)}{\Gamma(c)\Gamma(n-c+1)}$$

where $c \in \mathcal{C}_x$ indicates summation over the multiset $\mathcal{C}_x$ so elements with multiplicity m are added m times.[24]

For example, there are $r_{15} = \binom{6-1}{1-1} + \binom{6-1}{2-1} + \binom{6-1}{2-1} + \binom{6-1}{3-1} = \binom{5}{0} + \binom{5}{1} + \binom{5}{1} + \binom{5}{2} = 1 + 5 + 5 + 10 = 21$ distinct 6-dice rolls with an inverted sum of 15 (the green numbers are seen in the $\mathcal{C}_x$-column of Table 15).

As we did in section 2.5, let us regard distinct rolls as a function $R: \mathcal{O}_{6,n} \longrightarrow \mathcal{R}_{6,n}$. Let $R(O)$ denote the value of R applied to an argument $O \in \mathcal{O}_{6,n}$. Now, for each $S \epsilon \mathcal{S}_{6,n}$, let $o_S = |\{O \in \mathcal{O}_{6,n} : \mathrm{Supp}(R(O)) = S\}|$. In plain words, o_S is the number of outcomes corresponding to each S, i.e., the number of sequences O such that all elements in S, and no other values, are terms in O. For each $S \epsilon \mathcal{S}_{6,n}$, finding o_S is equivalent to finding the cardinality of the set of all *surjective* mappings from a set with $|O| = n$ elements onto S.[25] That cardinality is given by

$$o_S = |S|! \left\{ {n \atop |S|} \right\} = \sum_{k=0}^{|S|} (-1)^{|S|-k} \binom{|S|}{k} k^n = |S|! \sum_{k=0}^{|S|} \frac{(-1)^{|S|-k} k^n}{(|S|-k)!\, k!}$$

where the brackets $\left\{ {n \atop |S|} \right\}$ denotes the Stirling number of the second kind.[26]

With this, we can write $o_x = |\{O \in \mathcal{O}_{6,n} : \dot{\Sigma}(O) = x\}|$ as

$$o_x = \sum_{c \in \mathcal{C}_x} \sum_{k=0}^{c} (-1)^{c-k} \binom{c}{k} k^n$$

which gives us an expression for the probability of getting the inverted sum x with a roll of n standard dice:

$$\Pr(n, \dot{\Sigma}(R) = x) = \frac{o_x}{6^n} = \frac{1}{6^n} \sum_{c \in \mathcal{C}_x} \sum_{k=0}^{c} (-1)^{c-k} \binom{c}{k} k^n$$

$$\text{where } \mathcal{C}_x = \begin{cases} \{6\} & \text{for } x = 0 \\ \{5\} & \text{for } x = 1 \text{ or } x = 2 \\ \{4,5\} & \text{for } x = 3 \text{ or } x = 4 \\ \{4,4,5\} & \text{for } x = 5 \\ \{3,4,4,5\} & \text{for } x = 6 \\ \{3,4,4,4\} & \text{for } x = 7 \\ \{3,3,4,4\} & \text{for } x = 8 \\ \{3,3,3,4,4\} & \text{for } x = 9 \\ \{2,3,3,3,4\} & \text{for } x = 10 \text{ or } x = 11 \\ \{2,2,3,3,3\} & \text{for } x = 12 \\ \{2,2,3,3\} & \text{for } x = 13 \\ \{2,2,2,3\} & \text{for } x = 14 \\ \{1,2,2,3\} & \text{for } x = 15 \\ \{1,2,2\} & \text{for } x = 16 \\ \{1,2\} & \text{for } x = 17 \text{ or } x = 18 \\ \{1\} & \text{for } x = 19 \text{ or } x = 20 \end{cases}$$

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[24] $\Gamma(n) = (n-1)!$ is the *gamma function*. See https://en.wikipedia.org/wiki/Gamma_function for more information.

[25] See https://en.wikipedia.org/wiki/Surjective_function#Space_of_surjections.

[26] See https://en.wikipedia.org/wiki/Stirling_numbers_of_the_second_kind.

These sixteen conditions in the expression for $\mathcal{C}_x$, is as simple as it gets in this introductory text. It might very well be possible to deduct a simpler formula for $\Pr\big(n, \dot{\Sigma}(R) = x\big)$, but this is left for the reader as a challenging exercise that involves both the *partition function* $Q(n)$, a bivariate *generating function*, and *Gaussian binomial coefficients*.[27] For each x, each element c in the multiset $\mathcal{C}_x$ correspond to a set S (we have $c = |S|$), where $\dot{\Sigma}(S) = x$. Since $x = \Sigma(D) - \Sigma(S)$ with $D = \{1, 2, 3, 4, 5, 6\}$, we have $\Sigma(S) = 21 - x$. In other words, each element in $\mathcal{C}_x$ is the number of parts in a *strict* partition (the parts are *distinct*) of the number $21 - x$ with at most $|D| = 6$ parts.[28]

I refer to the references and the link section for a demonstration of the methods used for solving problems involving this type of partition restrictions. The appendix contains Python scripts that might also be useful.

3. The f-faced case

The *faces* ⚀, ⚁, ⚂, ⚃, ⚄, ⚅ of a standard die are the square *sides* of a cube. While there exist a limited number of regular-shaped dice (most commonly having 4, 8, 10, 12, or 20 sides), the concept of f-faced dice can easily be extended to include all theoretical situations, such as the enormous $f = 10^6$ or the boring $f = 1$.[29]

A *roll* is simply a randomly selected value from the set $D_f = \{1, 2, \ldots, f\}$, and the probability of each value being selected is $\frac{1}{f}$. The probability of getting the (regular) sum x in a roll with n f-faced dice is given by

$$\Pr(n, f, \Sigma(R) = x) = \frac{1}{f^n} \sum_{k=0}^{\left\lfloor \frac{x-n}{f} \right\rfloor} (-1)^k \binom{n}{k} \binom{x - kf - 1}{n - 1}$$

where $\lfloor\ \rfloor$ is the floor function (Uspensky, 1937). The proof of this uses the technique involving a generating function mentioned in section 2.7. Finding a similar expression for $\Pr\big(n, f, \dot{\Sigma}(R) = x\big)$ might be challenging, maybe even impossible. However, it *is* possible to use

$$\Pr\big(n, f, \dot{\Sigma}(R) = x\big) = \frac{1}{f^n} \sum_{c \in \mathcal{C}_{f,x}} \sum_{k=0}^{c} (-1)^{c-k} \binom{c}{k} k^n$$

where $\mathcal{C}_{f,x}$ now corresponds to the inverted sum x with f*-faced* dice, but it is a rather tedious approach, since both the number of multisets in $\mathcal{C}_{f,x}$ and the elements in each $c \in \mathcal{C}_{f,x}$ depend on f. A few examples are included below, for inspirational purposes. A Python script for calculating $\mathcal{C}_{f,x}$ can be found in the appendix.

$$\mathcal{C}_{3,x} = \begin{cases} \{3\} & \text{for } x = 0 \\ \{2\} & \text{for } x = 1 \text{ or } x = 2 \\ \{1,2\} & \text{for } x = 3 \\ \{1\} & \text{for } x = 4 \text{ or } x = 5 \end{cases}$$

$$\mathcal{C}_{4,x} = \begin{cases} \{4\} & \text{for } x = 0 \\ \{3\} & \text{for } x = 1 \text{ or } x = 2 \\ \{2,3\} & \text{for } x = 3 \text{ or } x = 4 \\ \{2,2\} & \text{for } x = 5 \\ \{1,2\} & \text{for } x = 6 \text{ or } x = 7 \\ \{1\} & \text{for } x = 8 \text{ or } x = 9 \end{cases}$$

$$\mathcal{C}_{5,x} = \begin{cases} \{5\} & \text{for } x = 0 \\ \{4\} & \text{for } x = 1 \text{ or } x = 2 \\ \{3,4\} & \text{for } x = 3 \text{ or } x = 4 \\ \{3,3,4\} & \text{for } x = 5 \\ \{2,3,3\} & \text{for } x = 6 \text{ or } x = 7 \\ \{2,2,3\} & \text{for } x = 8 \text{ or } x = 9 \\ \{1,2,2\} & \text{for } x = 10 \\ \{1,2\} & \text{for } x = 11 \text{ or } x = 12 \\ \{1\} & \text{for } x = 13 \text{ or } x = 14 \end{cases}$$

$$\mathcal{C}_{7,x} = \begin{cases} \{7\} & \text{for } x = 0 \\ \{6\} & \text{for } x = 1 \text{ or } x = 2 \\ \{5,6\} & \text{for } x = 3 \text{ or } x = 4 \\ \{5,5,6\} & \text{for } x = 5 \\ \{4,5,5,6\} & \text{for } x = 6 \\ \{4,5,5,5,6\} & \text{for } x = 7 \\ \{4,4,5,5,5\} & \text{for } x = 8 \\ \{4,4,4,5,5,5\} & \text{for } x = 9 \\ \{3,4,4,4,4,5,5\} & \text{for } x = 10 \text{ or } x = 11 \\ \{3,3,4,4,4,4,4,5\} & \text{for } x = 12 \\ \{3,3,3,4,4,4,4,5\} & \text{for } x = 13 \\ \{3,3,3,3,4,4,4,4\} & \text{for } x = 14 \\ \{2,3,3,3,3,4,4,4\} & \text{for } x = 15 \\ \{2,3,3,3,3,3,4,4\} & \text{for } x = 16 \\ \{2,2,3,3,3,3,4\} & \text{for } x = 17 \text{ or } x = 18 \\ \{2,2,2,3,3,3\} & \text{for } x = 19 \\ \{2,2,2,3,3\} & \text{for } x = 20 \\ \{1,2,2,2,3\} & \text{for } x = 21 \\ \{1,2,2,3\} & \text{for } x = 22 \\ \{1,2,2\} & \text{for } x = 23 \\ \{1,2\} & \text{for } x = 24 \text{ or } x = 25 \\ \{1\} & \text{for } x = 26 \text{ or } x = 27 \end{cases}$$

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27 See https://mathworld.wolfram.com/PartitionFunctionQ.html for information about $Q(n)$.
See https://en.wikipedia.org/wiki/Generating_function for information about generating functions.
See https://en.wikipedia.org/wiki/Gaussian_binomial_coefficient for information about Gaussian binomial coefficients.

28 See https://en.wikipedia.org/wiki/Partition_(number_theory) for information about partitions.

29 See https://en.wikipedia.org/wiki/Dice for more information on the various physical shapes of dice.

$$
C_{10,x} = \begin{cases}
\{10\} & \text{for } x = 0 \\
\{9\} & \text{for } x = 1 \text{ or } x = 2 \\
\{8,9\} & \text{for } x = 3 \text{ or } x = 4 \\
\{8,8,9\} & \text{for } x = 5 \\
\{7,8,8,9\} & \text{for } x = 6 \\
\{7,8,8,8,9\} & \text{for } x = 7 \\
\{7,7,8,8,8,9\} & \text{for } x = 8 \\
\{7,7,7,8,8,8,8,9\} & \text{for } x = 9 \\
\{6,7,7,7,7,8,8,8,8,9\} & \text{for } x = 10 \\
\{6,7,7,7,7,7,8,8,8,8,8\} & \text{for } x = 11 \\
\{6,6,7,7,7,7,7,7,7,8,8,8,8\} & \text{for } x = 12 \\
\{6,6,6,7,7,7,7,7,7,7,7,8,8,8,8\} & \text{for } x = 13 \\
\{6,6,6,6,6,7,7,7,7,7,7,7,7,7,8,8,8\} & \text{for } x = 14 \\
\{5,6,6,6,6,6,6,7,7,7,7,7,7,7,7,7,7,8,8,8\} & \text{for } x = 15 \\
\{5,6,6,6,6,6,6,6,6,6,7,7,7,7,7,7,7,7,7,7,8,8\} & \text{for } x = 16 \\
\{5,5,6,6,6,6,6,6,6,6,6,6,7,7,7,7,7,7,7,7,7,7,8,8\} & \text{for } x = 17 \\
\{5,5,5,6,6,6,6,6,6,6,6,6,6,6,6,6,7,7,7,7,7,7,7,7,7,7,8\} & \text{for } x = 18 \\
\{5,5,5,5,5,6,6,6,6,6,6,6,6,6,6,6,6,6,6,7,7,7,7,7,7,7,7,7,8\} & \text{for } x = 19 \\
\{5,5,5,5,5,5,5,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,7,7,7,7,7,7,7,7\} & \text{for } x = 20 \\
\{4,5,5,5,5,5,5,5,5,5,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,7,7,7,7,7,7,7\} & \text{for } x = 21 \\
\{4,5,5,5,5,5,5,5,5,5,5,5,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,7,7,7,7,7\} & \text{for } x = 22 \\
\{4,4,5,5,5,5,5,5,5,5,5,5,5,5,5,5,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,7,7,7,7\} & \text{for } x = 23 \\
\{4,4,4,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,7,7,7\} & \text{for } x = 24 \\
\{4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,6,6,6,6,6,6,6,6,6,6,6,6,6,6,7,7\} & \text{for } x = 25 \\
\{4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,6,6,6,6,6,6,6,6,6,6,6,6,6,7\} & \text{for } x = 26 \\
\{4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,6,6,6,6,6,6,6,6,6,6,7\} & \text{for } x = 27 \\
\{3,4,4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,6,6,6,6,6,6,6,6,6\} & \text{for } x = 28 \\
\{3,4,4,4,4,4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,6,6,6,6,6,6\} & \text{for } x = 29 \\
\{3,3,4,4,4,4,4,4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,6,6,6,6,6\} & \text{for } x = 30 \\
\{3,3,3,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,6,6,6\} & \text{for } x = 31 \\
\{3,3,3,3,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,5,5,5,6,6\} & \text{for } x = 32 \\
\{3,3,3,3,3,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,5,5,6\} & \text{for } x = 33 \\
\{3,3,3,3,3,3,3,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5,5,5,6\} & \text{for } x = 34 \\
\{3,3,3,3,3,3,3,3,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,5\} & \text{for } x = 35 \\
\{2,3,3,3,3,3,3,3,3,3,4,4,4,4,4,4,4,4,4,4,4,4,4,4,5,5,5,5,5\} & \text{for } x = 36 \\
\{2,3,3,3,3,3,3,3,3,3,3,4,4,4,4,4,4,4,4,4,4,4,4,4,5,5,5\} & \text{for } x = 37 \\
\{2,2,3,3,3,3,3,3,3,3,3,3,4,4,4,4,4,4,4,4,4,4,5,5\} & \text{for } x = 38 \\
\{2,2,3,3,3,3,3,3,3,3,3,3,4,4,4,4,4,4,4,4,4,5\} & \text{for } x = 39 \\
\{2,2,2,3,3,3,3,3,3,3,3,3,3,4,4,4,4,4,4,5\} & \text{for } x = 40 \\
\{2,2,2,3,3,3,3,3,3,3,3,3,4,4,4,4,4\} & \text{for } x = 41 \\
\{2,2,2,2,3,3,3,3,3,3,3,3,4,4,4\} & \text{for } x = 42 \\
\{2,2,2,2,3,3,3,3,3,3,3,4,4\} & \text{for } x = 43 \\
\{2,2,2,2,2,3,3,3,3,3,4\} & \text{for } x = 44 \\
\{1,2,2,2,2,3,3,3,3,4\} & \text{for } x = 45 \\
\{1,2,2,2,2,3,3,3\} & \text{for } x = 46 \\
\{1,2,2,2,3,3\} & \text{for } x = 47 \\
\{1,2,2,2,3\} & \text{for } x = 48 \\
\{1,2,2,3\} & \text{for } x = 49 \\
\{1,2,2\} & \text{for } x = 50 \\
\{1,2\} & \text{for } x = 51 \text{ or } x = 52 \\
\{1\} & \text{for } x = 53 \text{ or } x = 54
\end{cases}
$$

The expression for $C_{10,x}$ clearly reveals the need for generating functions. Ten-sided dice are popular and used in many different games, so we'll look a bit closer at the case $f = 10$ in the next section.

In general, with dice numbered from 1 to f, the number of possible inverted sums using n dice is given by

$$\dot{\sigma}_f(n) = \sum_{k=1}^{f} k \cdot [n + k > f]$$

and the inverted sums range from $\frac{f(f+1)}{2} - \dot{\sigma}_f(n)$ to $\frac{f(f+1)}{2} - 1$. For instance, with four ten-sided dice, we have $\dot{\sigma}_{10}(4) = 34$ possible inverted sums (ranging from 21 to 54). When $n = f - 1$, the inverted sums range from 1 to $\frac{f(f+1)}{2} - 1$. When $n \geq f$, the inverted sums range from 0 to $\frac{f(f+1)}{2} - 1$.

3.1. Using ten-faced dice

With the notation introduced earlier, let $\Sigma[\mathcal{R}_{f,n}]$ denote the image of $\mathcal{R}_{\mathrm{f,n}}$ under Σ, in short, the image of Σ. It is easy to see that $\Sigma[\mathcal{R}_{f,n}] = \{n, n+1, n+2, \ldots, fn\}$ and $\sigma_f(n) = |\Sigma[\mathcal{R}_{f,n}]| = (f-1)n+1$. With $f = 10$, we have $\Sigma[\mathcal{R}_{10,n}] = \{n, n+1, n+2, \ldots, 10n\}$ and $\sigma_{10}(n) = 9n+1$.

Let $\dot{\Sigma}[\mathcal{R}_{10,n}]$ denote the image of $\dot{\Sigma}$. The table below contains the intersection $\Sigma[\mathcal{R}_{10,n}] \cap \dot{\Sigma}[\mathcal{R}_{10,n}]$ and the union $\Sigma[\mathcal{R}_{10,n}] \cup \dot{\Sigma}[\mathcal{R}_{10,n}]$ for all n. It is informative to compare this with Table 14.

n	$\Sigma[\mathcal{R}_{10,n}] \cap \dot{\Sigma}[\mathcal{R}_{10,n}]$	$\Sigma[\mathcal{R}_{10,n}] \cup \dot{\Sigma}[\mathcal{R}_{10,n}]$	n	$\Sigma[\mathcal{R}_{10,n}] \cap \dot{\Sigma}[\mathcal{R}_{10,n}]$	$\Sigma[\mathcal{R}_{10,n}] \cup \dot{\Sigma}[\mathcal{R}_{10,n}]$
1	$\emptyset$	$D_{10} \cup \{45, 46, \ldots, 54\}$	29	$\{29, 30, \ldots, 54\}$	$\{0, 1, \ldots, 290\}$
2	$\emptyset$	$\{2, \ldots, 20\} \cup \{36, \ldots, 54\}$	30	$\{30, 31, \ldots, 54\}$	$\{0, 1, \ldots, 300\}$
3	$\{28, 29, 30\}$	$\{3, 4, \ldots, 54\}$	31	$\{31, 32, \ldots, 54\}$	$\{0, 1, \ldots, 310\}$
4	$\{21, 22, \ldots, 40\}$	$\{4, 5, \ldots, 54\}$	32	$\{32, 33, \ldots, 54\}$	$\{0, 1, \ldots, 320\}$
5	$\{15, 16, \ldots, 50\}$	$\{5, 6, \ldots, 54\}$	33	$\{33, 34, \ldots, 54\}$	$\{0, 1, \ldots, 330\}$
6	$\{10, 11, \ldots, 54\}$	$\{6, 7, \ldots, 60\}$	34	$\{34, 35, \ldots, 54\}$	$\{0, 1, \ldots, 340\}$
7	$\{7, 8, \ldots, 54\}$	$\{6, 7, \ldots, 70\}$	35	$\{35, 36, \ldots, 54\}$	$\{0, 1, \ldots, 350\}$
8	$\{8, 9, \ldots, 54\}$	$\{3, 4, \ldots, 80\}$	36	$\{36, 37, \ldots, 54\}$	$\{0, 1, \ldots, 360\}$
9	$\{9, 10, \ldots, 54\}$	$\{1, 2, \ldots, 90\}$	37	$\{37, 38, \ldots, 54\}$	$\{0, 1, \ldots, 370\}$
10	$\{10, 11, \ldots, 54\}$	$\{0, 1, \ldots, 100\}$	38	$\{38, 39, \ldots, 54\}$	$\{0, 1, \ldots, 380\}$
11	$\{11, 12, \ldots, 54\}$	$\{0, 1, \ldots, 110\}$	39	$\{39, 40, \ldots, 54\}$	$\{0, 1, \ldots, 390\}$
12	$\{12, 13, \ldots, 54\}$	$\{0, 1, \ldots, 120\}$	40	$\{40, 41, \ldots, 54\}$	$\{0, 1, \ldots, 400\}$
13	$\{13, 14, \ldots, 54\}$	$\{0, 1, \ldots, 130\}$	41	$\{41, 42, \ldots, 54\}$	$\{0, 1, \ldots, 410\}$
14	$\{14, 15, \ldots, 54\}$	$\{0, 1, \ldots, 140\}$	42	$\{42, 43, \ldots, 54\}$	$\{0, 1, \ldots, 420\}$
15	$\{15, 16, \ldots, 54\}$	$\{0, 1, \ldots, 150\}$	43	$\{43, 44, \ldots, 54\}$	$\{0, 1, \ldots, 430\}$
16	$\{16, 17, \ldots, 54\}$	$\{0, 1, \ldots, 160\}$	44	$\{44, 45, \ldots, 54\}$	$\{0, 1, \ldots, 440\}$
17	$\{17, 18, \ldots, 54\}$	$\{0, 1, \ldots, 170\}$	45	$\{45, 46, \ldots, 54\}$	$\{0, 1, \ldots, 450\}$
18	$\{18, 19, \ldots, 54\}$	$\{0, 1, \ldots, 180\}$	46	$\{46, 47, \ldots, 54\}$	$\{0, 1, \ldots, 460\}$
19	$\{19, 20, \ldots, 54\}$	$\{0, 1, \ldots, 190\}$	47	$\{47, 48, \ldots, 54\}$	$\{0, 1, \ldots, 470\}$
20	$\{20, 21, \ldots, 54\}$	$\{0, 1, \ldots, 200\}$	48	$\{48, 49, \ldots, 54\}$	$\{0, 1, \ldots, 480\}$
21	$\{21, 22, \ldots, 54\}$	$\{0, 1, \ldots, 210\}$	49	$\{49, 50, \ldots, 54\}$	$\{0, 1, \ldots, 490\}$
22	$\{22, 23, \ldots, 54\}$	$\{0, 1, \ldots, 220\}$	50	$\{50, 51, \ldots, 54\}$	$\{0, 1, \ldots, 500\}$
23	$\{23, 24, \ldots, 54\}$	$\{0, 1, \ldots, 230\}$	51	$\{51, 52, 53, 54\}$	$\{0, 1, \ldots, 510\}$
24	$\{24, 25, \ldots, 54\}$	$\{0, 1, \ldots, 240\}$	52	$\{52, 53, 54\}$	$\{0, 1, \ldots, 520\}$
25	$\{25, 26, \ldots, 54\}$	$\{0, 1, \ldots, 250\}$	53	$\{53, 54\}$	$\{0, 1, \ldots, 530\}$
26	$\{26, 27, \ldots, 54\}$	$\{0, 1, \ldots, 260\}$	54	$\{54\}$	$\{0, 1, \ldots, 540\}$
27	$\{27, 28, \ldots, 54\}$	$\{0, 1, \ldots, 270\}$	$\boldsymbol{n > 54}$	$\emptyset$	$\{0, 1, \ldots, 54\} \cup \{n, n+1, \ldots, 10n\}$
28	$\{28, 29, \ldots, 54\}$	$\{0, 1, \ldots, 280\}$			

Table 16. Intersections and unions of the images $\Sigma[\mathcal{R}_{10,n}]$ and $\dot{\Sigma}[\mathcal{R}_{10,n}]$.

The case $n = 2$ has an interesting union $\Sigma[\mathcal{R}_{10,2}] \cup \dot{\Sigma}[\mathcal{R}_{10,2}]$ consisting of two disjoint sets with 19 elements each. This could maybe be relevant in some game yet to be invented. But inverted sums of two-dice rolls are rather boring. The case $n = 9$, where $\dot{\Sigma}[\mathcal{R}_{10,n}] = \{1, 2, \ldots, 54\}$ and $\Sigma[\mathcal{R}_{10,n}] \cup \dot{\Sigma}[\mathcal{R}_{10,n}] = \{1, 2, \ldots, 90\}$ is pretty, but to play with *nine* ten-sided dice is impractical. Conclusively, Table 16 indicates that using inverted dice sums with ten-sided dice might not the most interesting option for game makers. However, there could still be some mathematical significance to be discovered.

As an example, the probability table for a roll with three 10-faced dice is given below (the appendix contains Python code for calculating probabilities for n f-sided dice). It is left to the reader to find solutions to the equation $\dot{\Sigma}(\mathrm{R}) = \Sigma(\mathrm{R})$ discussed in earlier sections.

When it comes to solutions to the equation $\Pr(n, 10, \dot{\Sigma}(R) = x) = \Pr(n, 10, \Sigma(R) = x)$, for rolls with a maximum of 50 dice, we have the two non-trivial solutions $\Pr(3, 10, \dot{\Sigma}(R) = 28) = \Pr(3, 10, \Sigma(R) = 28) = 0.6\%$ and $\Pr(4, 10, \dot{\Sigma}(R) = 29) = \Pr(4, 10, \Sigma(R) = 29) = 0.0348\%$. I leave it to the reader to find the corresponding rolls.

x	$\Pr(3, 10, \Sigma(R) = x)$	x	$\Pr(3, 10, \dot{\Sigma}(R) = x)$
3	1/1000	28	6/1000
4	3/1000	29	6/1000
5	6/1000	30	12/1000
6	10/1000	31	18/1000
7	15/1000	32	24/1000
8	21/1000	33	30/1000
9	28/1000	34	42/1000
10	36/1000	35	48/1000
11	45/1000	36	60/1000
12	55/1000	37	66/1000
13	63/1000	38	72/1000
14	69/1000	39	72/1000
15	73/1000	40	78/1000
16	75/1000	41	72/1000
17	75/1000	42	72/1000
18	73/1000	43	66/1000
19	69/1000	44	60/1000
20	63/1000	45	49/1000
21	55/1000	46	43/1000
22	45/1000	47	31/1000
23	36/1000	48	25/1000
24	28/1000	49	19/1000
25	21/1000	50	13/1000
26	15/1000	51	7/1000
27	10/1000	52	7/1000
28	6/1000	53	1/1000
29	3/1000	54	1/1000
30	1/1000		

Table 17. Possible sums and inverted sums of a three-dice roll (10-sided dice), with corresponding probabilities.

Comparing the table above with Table 8 shows that for 10-sided 3-dice rolls, we get the same near-symmetry for inverted sums around $x = 40$ as with 6-sided 3-dice rolls around $x = 12$. Symmetric probability values are marked in green and near-symmetric values are marked in orange above. The blue values belong to $\Sigma[\mathcal{R}_{10,3}] \cap \dot{\Sigma}[\mathcal{R}_{10,3}]$.

4. A set-theoretical generalisation

The concept of inverted sums can be applied to all multisets in a *finite* universe, i.e., not only multisets that relate to dice rolls. Let U (a set) be the finite universe of a *multiset* M with the support S.[30] This means that $S \subseteq U$ and $x \in S \Leftrightarrow x \in M$. Let $M^{\mathrm{C}} = \{x \in U : x \notin M\}$ and let $S^{\mathrm{C}} = \{x \in U : x \notin S\}$. Then $M^{\mathrm{C}} = S^{\mathrm{C}}$. Let $\Sigma(S)$ be the sum of all elements in S (with respect to some addition operator $+$), i.e.,

$$\Sigma(S) = \sum_{x \in S} x$$

Now, we define the *inverted sum* of M as

$$\dot{\Sigma}(M) = \sum_{x \in M^{\mathrm{C}}} x = \Sigma(U) - \Sigma(S)$$

which implies that $\dot{\Sigma}(M) = \dot{\Sigma}(S)$. When M is a *set* (or a *multiset* with all elements having the multiplicity 1), we have $\dot{\Sigma}(M) = \Sigma(U) - \Sigma(M)$. When $M^{\mathrm{C}} = \emptyset$, we have $\Sigma(M) \geq \Sigma(U)$ and $\dot{\Sigma}(M) = 0$.

[30] In texts concerning multisets, the universe of a multiset is often $\mathbb{N}$, but that is not the case in the situation presented here.

If S_1 and S_2 are two sets such that $S_1 = \text{Supp}(M_1)$ and $S_2 = \text{Supp}(M_2)$ for two multisets M_1 and M_2 in the same universe U, then $\dot{\Sigma}(S_1 \cap S_2) = \Sigma(U) - \Sigma(S_1 \cap S_2)$ and $\dot{\Sigma}(S_1 \cup S_2) = \Sigma(U) - \Sigma(S_1 \cup S_2)$. I encourage the reader to further investigate the identities that can be deducted from this.

As a final remark, I will mention that if U is allowed to be an *infinite* set with both a *lower bound* and an *upper bound* (such as $U = [a, b]$, where $a, b \in \mathbb{R}$ and $a < b$), then we can define $\dot{\Sigma}(S)$ for each $S \subseteq U$ with the help of integrals, where $\int_a^b x\,dx$ is the integral analogue of the sum $\Sigma(U)$. I leave it to the reader to investigate this.

5. About this paper

If you have read this far, you might have noticed that this text does not contain a lot of what could be described as *real math*. There are no formal definitions, theorems, corollaries etc., and that structure is chosen purposely, since the text is, above all, meant to be *inspirational*. I have attempted to use the correct terminology from the subfields of mathematics that the text refers to (mainly set theory, partition theory, and probability theory). The main target audience consists of dice game developers, mathematicians, and maybe also computer scientists.

The concept of "inverted dice sums" is relatively new (I invented it in 2013 in connection with the development of the game Inverted Dice™), and there is much yet to be written on the subject. However, I do not have the time to pursue it any further. My hope is that somebody else will treat it with the theoretical stringency it deserves.

The content has not undergone peer review (nor has it been proofread), as I am currently not affiliated with any mathematical community. Consequently, it is reasonable to assume that I may have made a few mistakes. I hope that these mistakes are of minimal significance and do not overshadow the ideas presented here. I encourage my readers to reach out to me if they come across any irregularities in the text, and particularly if they can contribute with more simple versions of the probability formulas. My e-mail address is information@simonjensen.com.

This is version 1.1 of this paper, published the 4th of August 2023 at my website www.simonjensen.com. If any typographical errors, logical errors, or other kinds of flaws are discovered, the text will be updated. Maybe, at some point, I will include more information about generating functions. The latest version of this paper will always be available at https://www.simonjensen.com/pdf/A_short_introduction_to_the_theory_of_inverted_dice_sums.pdf.

6. References

Uspensky, J. V., *Introduction to Mathematical Probability*. McGraw-Hill, New York (1937), pp. 23–24.

Knopfmacher, A., Robbins, N., Identities for the total number of parts in partitions of integers, Utilitas Mathematica 67 (2005), pp. 9–18.

Bodlaender, H. L., Jansen, K., Restrictions of graph partition problems. Part I, Elsevier B.V. (1995), DOI: 10.1016/0304-3975(95)00057-4.

7. Links

https://mathworld.wolfram.com/Dice.html

https://en.wikipedia.org/wiki/Multiset

https://en.wikipedia.org/wiki/Sequence

https://en.wikipedia.org/wiki/Family_of_sets

https://en.wikipedia.org/wiki/Power_set

https://en.wikipedia.org/wiki/Gamma_function

https://en.wikipedia.org/wiki/Surjective_function#Space_of_surjections

https://en.wikipedia.org/wiki/Stirling_numbers_of_the_second_kind

https://mathworld.wolfram.com/PartitionFunctionQ.html

https://oeis.org/A000009

https://en.wikipedia.org/wiki/Generating_function

https://mathworld.wolfram.com/GeneratingFunction.html

https://en.wikipedia.org/wiki/Gaussian_binomial_coefficient

https://en.wikipedia.org/wiki/Partition_(number_theory)

https://www.simonjensen.com/InvertedDice

https://en.wikipedia.org/wiki/Dice

Appendix

The Python code below can be used to calculate the probabilities for inverted dice sums.

```
from math import comb #comb(n,k) gives binomal coefficient "n choose k"
from math import floor
from itertools import chain, combinations

def powerset_plus(iterable): #list(powerset_plus(A)) produces list of all non-empty subsets of A (as tuples)
    s = list(iterable)
    return chain.from_iterable(combinations(s, r) for r in range(1, len(s) + 1))

def probabilities(f,n):
    o_x=list([] for _ in range(0, sum_D))
    o_y=list([] for _ in range(0, f*n+1))
    print("\nProbabilities for inverted sums, Pr(f,n,Ṡigma(R)=x):")
    for x in range(0, sum(D)):
        o_x[x]=0
        for c in C_x[x]:
            for i in range(0, c + 1):
                o_x[x]+=(-1)**(c-i)*comb(c,i)*(i**n)
        if not o_x[x]==0:
            print(f"Pr({n},{f},Ṡigma(R)={x}) = {o_x[x]}/{f**n} ≈
{int((10**(n+2))*o_x[x]/(f**n))/(10**n)}%")
    print("\nProbabilities for regular sums, Pr(n,f,Sigma(R)=x):")
    for y in range(n, f*n+1):
        o_y[y]=0
        for a in range(0, floor((y-n)/f)+1):
            o_y[y]+=(-1)**(a)*comb(n,a)*comb(y-a*f-1,n-1)
        if not o_y[y]==0:
            print(f"Pr({n},{f},Sigma(R)={y}) = {o_y[y]}/{f**n} ≈
{int((10**(n+2))*o_y[y]/(f**n))/(10**n)}%")
    print(f"\nSolutions to Pr({n},{f},Ṡigma(R)=x) = Pr({n},{f},Sigma(R)=x):")
    no_solutions=True
    for x in range(0, sum_D):
        for y in range(n, f*n+1):
            if x==y and o_x[x]==o_y[y]:
                print(f"Pr({n},{f},Ṡigma(R)={x}) = Pr({n},{f},Sigma(R)={y}) = {o_x[x]}/{f**n}")
                no_solutions=False
    if no_solutions:
        print("None!")

f = int(input("Faces on each die (f): "))
N = int(input("Number of dice (n) (program will be is looping from 1 to n): "))
print(f"---------------------\nUsing {f}-faced dice):\n---------------------")
D = list(range(1, f + 1))
sum_D = sum(D)
print(f"D = {D}\nΣ(D)={sum_D}")
S_f_n = list(powerset_plus(D))
C_x=list([] for _ in range(0, sum_D))
print(f"\nS_{f} = {S_f_n}")
print(f"\n|S_{f}| = {len(S_f_n)}\n")
for S in S_f_n:
    C_x[sum_D-sum(S)].append(len(S))
for x in range(0, sum(D)):
    print(f"C_{f}_{x} = {C_x[x]}")
print("\n")
for n in range(1,N+1):
    print(f"\nRolls with {n} dice):\n--------------------")
    sigma_overdot_f=0
    for k in range(1,f+1):
        if n+k>f:
            sigma_overdot_f+=k
    print(f"Number of possible inverted sums: {sigma_overdot_f}")
    image_Sigma=set(range(n,f*n+1))
    image_Sigma_overdot=set(range(sum_D-sigma_overdot_f,sum_D))
    images_intersection=image_Sigma.intersection(image_Sigma_overdot)
    images_union=image_Sigma.union(image_Sigma_overdot)
    print(f"Intersection of images: {images_intersection}")
    print(f"Union of images: {images_union}")
    probabilities(f,n)
```